Representations of $SU(1,1)$ in Non-commutative Space Generated by the Heisenberg Algebra

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Abstract: $SU(1,1)$ is considered as the automorphism group of the Heisenberg algebra H. The basis in the Hilbert space K of functions on H on which the irreducible representations of the group are realized is explicitly constructed. The addition theorems are derived.

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1. Introduction

Investigating the properties of manifolds by means of the symmetries they admit has a long history. Non-commutative geometries have become the subject of similar studies in recent decades. For example there exists an extensive literature on the q-deformed groups $E_q(2)$ and $SU_q(2)$ which are the automorphism groups of the quantum plane $zz^* = qz^*z$ and the quantum sphere respectively [1]. Using group theoretical methods the invariant distance and the Green functions have also been written in these deformed spaces [2].

In the recent work we started to analyze yet another non-commutative space $[z, z^*] = 1$ (i. e. the space generated by the Heisenberg algebra) by means of its automorphism groups: We considered $E(2)$ group transformations in z, z^* space; and constructed the basis (which are written in terms of the Kummer functions) in this space where the unitary irreducible representations of $E(2)$ are realized [3]. This analysis revealed a peculiar connection between the 2-dimensional Euclidean group and the Kummer functions.

In the present work we continue to study the same non-commutative space $[z, z^*] = 1$, this time by means of the other admissible automorphism group $SU(1,1)$.

In Section 2 we define $SU(1,1)$ in the Heisenberg algebra H and construct the unitary representations of the group in the Hilbert space X where H is realized.

In Section 3 we classify the invariant subspaces in the space of the bounded functions on H where the irreducible representations of $SU(1,1)$ are realized.

In Section 4 we show that in the Hilbert space K of the square integrable functions only principal series is unitary. We construct the orthonormal basis in K which can be written in terms of the Jacobi functions.

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Section 5 is devoted to the addition theorems. These theorems provide a group theoretical interpretation for the already existing identities involving the hypergeometric functions which all are actually the Jacobi functions. They may also lead to new identities.

2. Weyl representations of $SU(1,1)$

The one dimensional Heisenberg algebra H is the 3-dimensional vector space with the basis elements $\{z, z^*, 1\}$ and the bilinear antisymmetric product

$$[z, z^*] = 1. \tag{1}$$

The $*$-representation of H in the suitable dense subspace of the Hilbert space X with the complete orthonormal basis $\{\mid n\rangle\}$, $n = 0, 1, 2, ...$ is given by

$$z \mid n\rangle = \sqrt{n} \mid n-1\rangle, \quad z^* \mid n\rangle = \sqrt{n+1} \mid n+1\rangle. \tag{2}$$

Let us represent the pseudo-unitary group $SU(1,1)$ in the vector space H

$$g\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\begin{pmatrix} z \\ z^* \end{pmatrix}. \tag{3}$$

Due to

$$a\overline{a} - b\overline{b} = 1 \tag{4}$$

the transformations (3) preserve the commutation relation

$$[gz, gz^*] = [z, z^*]. \tag{5}$$

Therefore

$$gz = U(g)zU^{-1}(g), \quad gz^* = U(g)z^*U^{-1}(g) \tag{6}$$

where $U(g)$ is the unitary representation of $SU(1,1)$ in X:

$$U(g_1)U(g_2) = U(g_1g_2), \quad U^*(g) = U^{-1}(g) = U(g^{-1}). \tag{7}$$

The Cartan decomposition for the group reads

$$g = k(\phi)h(\alpha)k(\psi), \tag{8}$$

where

$$k(\psi) = \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix}, \quad h(\alpha) = \begin{pmatrix} \cosh\frac{\alpha}{2} & \sinh\frac{\alpha}{2} \\ \sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}. \tag{9}$$

For the subgroup $k(\psi)$ we have

$$U(k(\psi)) \mid n\rangle = e^{-i\frac{n\psi}{2}} \mid n\rangle. \tag{10}$$

Let us choose the following realizations for z, z^* and X:

$$z = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \quad z^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}), \tag{11}$$

$$\langle x \mid n \rangle = \Psi_n(x), \quad \Psi_n(x) = \sqrt{\frac{e^{-x^2}}{2^n n! \sqrt{\pi}}} H_n(x), \tag{12}$$

where H_n is the Hermite polynomial. From

$$h(\alpha)z = \frac{1}{\sqrt{2}}(xe^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}} \frac{d}{dx}) \tag{13}$$

and

$$\int_{-\infty}^{\infty} dx \overline{\Psi_m(x)} \Psi_n(x) = \delta_{nm} \tag{14}$$

we get

$$U(h(\alpha))\Psi_m(x) = e^{\frac{\alpha}{4}} \Psi_m(e^{\frac{\alpha}{2}} x). \tag{15}$$

Matrix elements of $U(h(\alpha))$ in the basis $\{\mid n\rangle\}$ reads

$$U_{mn}(h) \equiv \langle m \mid U(h(\alpha)) \mid n \rangle = e^{\frac{\alpha}{4}} \int_{-\infty}^{\infty} dx \overline{\Psi_m(x)} \Psi_n(e^{\frac{\alpha}{2}} x). \tag{16}$$

Evaluating this integral we get

$$U_{mn}(h) = \frac{2^{\frac{m-n}{2}}}{(\frac{n-m}{2})!} \sqrt{\frac{n! \sinh^{n-m} \frac{\alpha}{2}}{m! \cosh^{n+m+1} \frac{\alpha}{2}}} F(-\frac{m}{2}, \frac{1-m}{2}; 1 + \frac{n-m}{2}; -\sinh^2 \frac{\alpha}{2}) \tag{17}$$

if $n \geq m$ and $n + m$ is even and

$$U_{mn}(h) = 0 \tag{18}$$

if $n + m$ is odd. For $m \geq n$ one has to replace m, n and α in the above formulas by n, m and $-\alpha$ respectively.

3. Irreducible representations of $SU(1,1)$ in H

The formula

$$T(g)F(z) = F(gz) \tag{19}$$

defines the representation of $SU(1,1))$ in the space K_0 of bounded operators in the Hilbert space X representable as the finite sums

$$F = \sum (f_n(\zeta) z^n + z^{*n} f_{-n}(\zeta)). \tag{20}$$

Here $f_n(\zeta)$ are functions of the self-adjoint operator $\zeta = z^* z$. Using (6) we can rewrite (19) in the form

$$T(g)F(z) = U(g)F(z)U^*(g) \tag{21}$$

With the one parameter subgroups $g_1 = h(\epsilon)$, $g_2 = k(\frac{\pi}{2})h(\epsilon)k(-\frac{\pi}{2})$ and $g_3 = k(\epsilon)$ of $SU(1,1)$ we associate the linear operators $E_k : K_0 \to K_0$

$$E_k(F) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(T(g_k)F - F) \tag{22}$$

with the limit being taken in the strong operator topology. Inserting (21) into (22) we get (with $H_\pm = -E_1 \mp iE_2, \quad H = iE_3$)

$$H_-(F) = \frac{1}{2}[F, z^2], \quad H_+(F) = \frac{1}{2}[z^{*2}, F], \quad H(F) = \frac{1}{2}[\zeta, F], \tag{23}$$

which implies the Lie algebra of $SU(1,1)$

$$[H_+, H_-] = 2H, \quad [H, H_\pm] = \pm H_\pm. \tag{24}$$

The irreducible representations labelled by pair (τ, ϵ), $\tau \in C$ and $\epsilon = 0, \frac{1}{2}$ are given by the formulas [4]

$$H_- D_k^{(\tau,\epsilon)} = -(k + \tau + \epsilon) D_{k-1}^{(\tau,\epsilon)}, \tag{25}$$

$$H_+ D_k^{(\tau,\epsilon)} = (k - \tau + \epsilon) D_{k+1}^{(\tau,\epsilon)}, \tag{26}$$

$$H D_k^{(\tau,\epsilon)} = (k + \epsilon) D_k^{(\tau,\epsilon)}. \tag{27}$$

(23) and (27) imply

$$D_k^{(\tau,\epsilon)} = z^{*2(k+\epsilon)} f_k^{(\tau,\epsilon)}(\zeta) \tag{28}$$

for $k \geq 0$ and

$$D_k^{(\tau,\epsilon)} = f_k^{(\tau,\epsilon)}(\zeta) z^{-2(k+\epsilon)} \tag{29}$$

for $k < 0$. By substituting (28) in (25) and (26) with

$$f_k^{(\tau,\epsilon)}(\zeta) = \sum_{n=0}^{\infty} \frac{(-)^n 2^{n+k+\epsilon}}{n!} C_{kn} z^{*n} z^n \tag{30}$$

we get the recurrence relations

$$nC_{kn-1} + \frac{k + \epsilon + \tau}{2k + 2\epsilon + n - 1} C_{k-1n} - (2k + 2\epsilon + n) C_{kn} = 0, \tag{31}$$

$$C_{kn+1} - C_{kn+2} - (k + \epsilon - \tau) C_{k+1n} = 0, \tag{32}$$

which are solved by

$$C_{kn} = \frac{\Gamma(1 + \tau + \epsilon + k + n)}{\Gamma(1 + 2\epsilon + 2k + n)}. \tag{33}$$

Using

$$z^{*n} z^n = \zeta(\zeta - 1)...(\zeta - n + 1) \tag{34}$$

for $k \geq 0$ we get

$$f_k^{(\tau,\epsilon)}(\zeta) = (-2)^{k'} \frac{\Gamma(1 + \tau + k')}{\Gamma(1 + 2k')} F(-\zeta, 1 + \tau + k'; 1 + 2k'; 2), \tag{35}$$

where $k' = k + \epsilon$. The functions $f_k^{(\tau,\epsilon)}$ for $k < 0$ is shown to be defined from the expression

$$f_k^{(\tau,\epsilon)}(\zeta) = f_{-k}^{(\tau,-\epsilon)}(\zeta). \tag{36}$$

From (25), (26) and (27) we conclude that $SU(1,1)$ admits the following irreducible representations:

i) $T_{(\tau,\epsilon)}: \quad (\tau+\epsilon)\overline{\in}Z$

ii) $T^{\pm}_{(\tau,\epsilon)}: \quad (\tau+\epsilon) \in Z,\ \tau - \epsilon < 0$, that is $\tau = -\frac{1}{2}, -1, -\frac{3}{2}, ...$

iii) $T^{0}_{(\tau,\epsilon)}: \quad (\tau+\epsilon) \in Z,\ \tau - \epsilon \geq 0$, that is $\tau = 0, \frac{1}{2}, 1, \frac{3}{2}...$

The corresponding invariant subspaces are:

i) $V_{(\tau,\epsilon)}$ generated by $\{D^{(\tau,\epsilon)})_k\}_{k=-\infty}^{\infty}$

ii) $V^{+}_{(\tau,\epsilon)}$ and $V^{-}_{(\tau,\epsilon)}$ generated by $\{D^{(\tau,\epsilon)})_k\}_{k=-\infty}^{\tau-\epsilon}$ and $\{D^{(\tau,\epsilon)})_k\}_{k=-\tau-\epsilon}^{\infty}$

iii) $V^{0}_{(\tau,\epsilon)}$ generated by $\{D^{(\tau,\epsilon)})_k\}_{k=-\tau-\epsilon}^{\tau-\epsilon}$

4. Unitary irreducible representations of $SU(1,1)$ in H

We can define the norm in the subspace of K_0 with $f_n(\zeta)$ in (20) being the functions with finite support in $Spect(\zeta) = \{0, 1, 2, ...\}$ as

$$|| F ||= \sqrt{tr(F^*F)}. \tag{37}$$

Completion of this subspace leads to the Hilbert space K of the square integrable functions in the linear space H with the scalar product

$$(F, G) = tr(F^*G). \tag{38}$$

Using (21), the unitarity of $U(g)$ and the property of the trace we conclude that the representation $T(g)$ in K is unitary. (23) implies the real structure in the Lie algebra

$$H^*_{\pm} = -H_{\mp}, \quad H^* = H. \tag{39}$$

To investigate the unitarity of the irreducible representations in the Hilbert space K classified in the previous section we consider the orthogonality condition for the basis elements $D_k^{(\tau,\epsilon)}$. Using (2) and (28) we get

$$(D_k^{(\tau,\epsilon)}, D_m^{(\tau',\epsilon')}) = \delta_{mk}\delta_{\epsilon\epsilon'}\sum_{n=0}^{\infty}\frac{(n+2k+2\epsilon)!}{n!}\overline{f_k^{(\tau,\epsilon)}(-n)}f_k^{(\tau',\epsilon)}(-n). \tag{40}$$

Putting

$$s = 1 - e^{-t}, \quad \lambda = 1 + 2(k+\epsilon) + \mu \tag{41}$$

in the formula [5]

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}s^n F(-n, a; \lambda; 2)F(-n, b; \lambda; 2) =$$

$$= (1-s)^{a+b-\lambda}(1+s)^{-a-b}F(a, b; \lambda; \frac{4s}{(1+s)^2}) \tag{42}$$

and taking first the limit $\mu \to +0$ and then $t \to \infty$ we obtain for $\tau = -\frac{1}{2}+i\rho$, $\rho \in R$ the orthogonality relations

$$(D_k^{(-\frac{1}{2}+i\rho,\epsilon)}, D_m^{(-\frac{1}{2}+i\rho',\epsilon')}) = \delta_{mk}\delta_{\epsilon\epsilon'}\delta(\rho-\rho'). \tag{43}$$

In the deriving of the above relation we used

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{44}$$

and the representation

$$\lim_{t\to\infty} \frac{e^{-izt}}{z+i0} = -2\pi i\delta(z) \tag{45}$$

for the Dirac delta function. For other values of τ there is no orthogonality condition. Thus in K only the representation $T_{(\tau,\epsilon)}$ with $\tau = -\frac{1}{2}+i\rho$ of Section 3, which is the principal series is unitary.

5. The addition theorems

(**i**) Restriction of (19) on the subspace $V_{(\tau,\epsilon)}$ reads:

$$T(g)D_k^{(\tau,\epsilon)} = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{46}$$

or

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{47}$$

where

$$t_{kn}^{(\tau,\epsilon)}(g) = \frac{e^{-i(k+\epsilon)\phi-i(k+\epsilon)\psi}}{(k-n)!}\frac{\Gamma(1+\tau-\epsilon-n)\sinh^{k-n}\frac{\alpha}{2}}{\Gamma(1+\tau-\epsilon-k)\cosh^{k+n+2\epsilon}\frac{\alpha}{2}}\times$$
$$\times F(-\tau-\epsilon-n, 1+\tau-\epsilon-n; 1+k-n; -\sinh^2\frac{\alpha}{2}) \tag{48}$$

are the matrix elements of the irreducible representations which are valid for $k \geq n$. For $k < n$ one has to replace k and n on the right hand side by $-k$ and $-n$ respectively.

(**ii**) Restriction of (19) on the subspaces $V^+_{(\tau,\epsilon)}$ and $V^-_{(\tau,\epsilon)}$ gives the following addition theorems:

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\infty}^{\tau-\epsilon} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{49}$$

and

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\tau-\epsilon}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)}. \tag{50}$$

(**iii**) On the subspaces $V^0_{(\tau,\epsilon)}$ the addition theorem reads

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\tau-\epsilon}^{\tau-\epsilon} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)}. \tag{51}$$

Sandwiching both sides of (47), (49), (50) and (51) between the states $\langle l \mid$ and $\mid s\rangle$ we get

$$\sum_{m,t=0}^{\infty} U_{lm}(g)\overline{U_{st}(g)}(D_k^{(\tau,\epsilon)})_{mt} = \sum_n t_{nk}^{(\tau,\epsilon)}(g)(D_n^{(\tau,\epsilon)})_{ls} \tag{52}$$

Multiplying (47), (49), (50) and (51) by $U(g)$ from the right and sandwiching them between the states $\langle l \mid$ and $\mid s\rangle$ we get

$$\sum_{m=0}^{\infty} U_{lm}(g)(D_k^{(\tau,\epsilon)})_{ms} = \sum_{m=0}^{\infty}\sum_n t_{nk}^{(\tau,\epsilon)}(g)(D_n^{(\tau,\epsilon)})_{lm}U_{ms}(g) \tag{53}$$

Multiplying (47), (49), (50) and (51) by $U^*(g)$ and $U(g)$ from the left and right respectively and sandwiching them between the states $\langle l \mid$ and $\mid s\rangle$ we get

$$(D_k^{(\tau,\epsilon)})_{ls} = \sum_{m,t=0}^{\infty}\sum_n t_{kn}^{(\tau,\epsilon)}(g)U_{ts}(g)\overline{U_{ml}(g)}(D_n^{(\tau,\epsilon)})_{mt} \tag{54}$$

where

$$(D_k^{(\tau,\epsilon)})_{mt} = \sqrt{\frac{m!}{t!}}f_k^{(\tau,\epsilon}(t)\delta_{m,t+2k+2\epsilon} \tag{55}$$

for $k \geq 0$ and

$$(D_k^{(\tau,\epsilon)})_{mt} = \sqrt{\frac{t!}{m!}}f_k^{(\tau,\epsilon}(m)\delta_{m,t+2k+2\epsilon} \tag{56}$$

for $k < 0$.

Finally we like to give two simple specific examples: Let $g = h(\alpha)$, $\epsilon, k = 0$ in (51) that is τ is positive integer. Taking $s, l = 0$ in (52) and (53) we get

$$P_\tau(\cosh\alpha) = \frac{1}{\sqrt{\pi}\cosh\frac{\alpha}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{n!}\tanh^{2n}\frac{\alpha}{2}F(-2n, 1+\tau; 1+n; -\sinh^2\frac{\alpha}{2}) \tag{57}$$

and

$$1 = \sum_{n=0}^{\tau}\frac{(-)^n!}{(n!)^2}\frac{(\tau+n)!}{(\tau-n)!}\tanh^{2n}\frac{\alpha}{2}F(-\tau, 1+\tau; 1+n; -\sinh^2\frac{\alpha}{2}) \tag{58}$$

respectively. P_τ in (57) is the Legendre function.

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